

On Beurling's sampling theorem in \mathbb{R}^n

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Abstract

We present an elementary proof of the classical Beurling sampling theorem which gives a sufficient condition for sampling of multi-dimensional band-limited functions.

1 Introduction

Let $\mathcal{S} \subset \mathbb{R}^n, n \geq 1$, be a compact. The Bernstein space $B_{\mathcal{S}}$ consists of all bounded functions on \mathbb{R}^n whose spectrum belongs to \mathcal{S} . The latter means that

$$\int_{\mathbb{R}^n} f(x) \hat{\varphi}(x) dx = 0, \quad f \in B_{\mathcal{S}},$$

for every smooth function $\varphi(x)$ whose support belongs to a ball disjoint from \mathcal{S} . Here $\hat{\varphi}$ denotes the Fourier transform

$$\hat{\varphi}(x) = \int_{\mathbb{R}^n} e^{-it \cdot x} \varphi(t) dt.$$

A set $\Lambda \subset \mathbb{R}^n$ is called a sampling set for $B_{\mathcal{S}}$, if there is a positive constant C such that

$$\|f\|_{\infty} \leq C \|f|_{\Lambda}\|_{\infty}, \quad \text{for every } f \in B_{\mathcal{S}},$$

where

$$\|f\|_{\infty} := \sup_{x \in \mathbb{R}^n} |f(x)|, \quad \|f|_{\Lambda}\|_{\infty} := \sup_{\lambda \in \Lambda} |f(\lambda)|.$$

It is a classical problem to determine when Λ constitutes a sampling set for $B_{\mathcal{S}}$. Beurling discovered the importance of the *lower uniform density* $D^-(\Lambda)$ of Λ for this problem:

$$D^-(\Lambda) := \lim_{r \rightarrow \infty} \frac{\min_{x \in \mathbb{R}^n} \text{Card}(\Lambda \cap (x + r\mathcal{B}))}{|r\mathcal{B}|},$$

where \mathcal{B} is the unit ball in \mathbb{R}^n , $x + r\mathcal{B}$ is the ball of radius r centered at x and $|\mathcal{S}|$ denotes the measure of a set \mathcal{S} . In [2] he proved the following

Theorem 1 *Let $\mathcal{S} = [a, b] \subset \mathbb{R}$. Then $\Lambda \subset \mathbb{R}$ is a sampling set for $B_{\mathcal{S}}$ if and only if*

$$D^-(\Lambda) > |\mathcal{S}|/2\pi. \quad (1)$$

Hence, when \mathcal{S} is an interval in \mathbb{R} , the sampling problem can be solved in terms of the density $D^-(\Lambda)$. Condition $D^-(\Lambda) \geq |\mathcal{S}|/(2\pi)^n$ remains *necessary* for sampling in $B_{\mathcal{S}}$, for every compact set $\mathcal{S} \subset \mathbb{R}^n$. This follows from a general result of Landau [6]. On the other hand, simple examples show that in dimension one condition (1) ceases to be *sufficient* already when \mathcal{S} is a union of two intervals.

A new phenomenon occurs in several dimensions: Even for the simplest sets \mathcal{S} like a ball or a cube, no sufficient conditions for sampling in $B_{\mathcal{S}}$ can be expressed in terms of $D^-(\Lambda)$. The reason for that is that the zeros of the multi-dimensional entire functions are not discrete. One can check that if $\mathcal{S} \subset \mathbb{R}^n$ contains at least two points, then there are functions $f \in B_{\mathcal{S}}$ whose zero set contains sets $\Lambda \subset \mathbb{R}^n$ with arbitrarily large $D^-(\Lambda)$. Clearly, if a function $f \in B_{\mathcal{S}}$ vanishes on Λ , then Λ is not a sampling set for $B_{\mathcal{S}}$ (see also discussion in [8], pp. 122–123).

In [1] Beurling obtained the following sufficient condition for sampling in $B_{\mathcal{B}}$:

Theorem 2 *Assume $\Lambda \subset \mathbb{R}^n$, $n \geq 1$, and $\rho < \frac{\pi}{2}$ satisfy*

$$\Lambda + \rho\mathcal{B} = \mathbb{R}^n.$$

Then

$$\|f\|_{\infty} \leq \frac{1}{1 - \sin \rho} \|f|_{\Lambda}\|_{\infty}, \quad \text{for every } f \in B_{\mathcal{B}}, \quad (2)$$

and so Λ is a sampling set for $B_{\mathcal{B}}$.

In fact, Beurling in [1] proves a result on balayage of Fourier–Stieltjes transforms which is equivalent to Theorem 2: *For every Dirac’s measure δ_{ξ} , there exists a finite measure with masses on Λ such that the values of their Fourier–Stieltjes transforms agree in the ball \mathcal{B} .* We use a completely different elementary approach which allows us to get a more general result, see Theorem 3 below. We

shall see that unlike the case of interpolation in several dimensions (see [7]), the "Beurling-type" sampling is in fact a one-dimensional phenomenon.

Observe that condition $\Lambda + \rho\mathcal{B} = \mathbb{R}^n$ in Theorem 2 means that Λ is an ρ -net, i.e. for every $x \in \mathbb{R}^n$ there exists $\lambda \in \Lambda$ with $|x - \lambda| \leq \rho$. Hence, every ρ -net with $\rho < \pi/2$ is a sampling set for $B_{\mathcal{B}}$. This is sharp: Beurling shows that the theorem ceases to be true for $\pi/2$ -nets.

Let us in what follows denote by \mathcal{K} a closed convex central-symmetric body with positive measure. Then

$$\mathcal{K}^\circ := \{x \in \mathbb{R}^n : x \cdot t \leq 1 \text{ for all } t \in \mathcal{K}\}$$

denotes the polar body of \mathcal{K} . In particular, we have $\mathcal{B}^\circ = \mathcal{B}$.

The following propositions are formulated in [1] without proof:

(i) Estimate (2) in Theorem 2 can be replaced with a better one:

$$\|f\|_\infty \leq \frac{1}{\cos \rho} \|f|_\Lambda\|_\infty. \quad (3)$$

(ii) Every set Λ satisfying $\Lambda + \rho\mathcal{K}^\circ = \mathbb{R}^n$ with some $\rho < \pi/2$ is a sampling set for $B_{\mathcal{K}}$.

We show that estimate (3) holds for every convex central-symmetric body \mathcal{K} :

Theorem 3 Assume $\Lambda \subset \mathbb{R}^n$ and $\rho < \frac{\pi}{2}$ satisfy

$$\Lambda + \rho\mathcal{K}^\circ = \mathbb{R}^n. \quad (4)$$

Then (3) is true, and so Λ is a sampling set for $B_{\mathcal{K}}$.

Clearly, condition (4) means that for every $x \in \mathbb{R}^n$ there exists $\lambda \in \Lambda$ such that $\|x - \lambda\|_{\mathcal{K}^\circ} \leq \rho$, where $\|x\|_{\mathcal{K}^\circ} := \inf_{a>0} \{x \in a\mathcal{K}^\circ\}$. Hence, every ρ -net in the norm $\|\cdot\|_{\mathcal{K}^\circ}$ is a sampling set for $B_{\mathcal{K}}$ provided $\rho < \pi/2$. This is sharp:

Proposition 1. Suppose a closed convex central-symmetric body \mathcal{S} contains a point x_0 with $\|x_0\|_{\mathcal{K}^\circ} = \pi/2$. Then there exists $\Lambda \subset \mathbb{R}^n$ with $\Lambda + \mathcal{S} = \mathbb{R}^n$ and a function $f \in B_{\mathcal{K}}$ such that $f(\lambda) = 0, \lambda \in \Lambda$.

Corollary 1. Suppose a closed convex central-symmetric body \mathcal{S} has the property that every set $\Lambda \subset \mathbb{R}^n$ satisfying $\Lambda + \mathcal{S} = \mathbb{R}^n$ is a sampling set for $B_{\mathcal{K}}$. Then $\mathcal{S} \subset \rho\mathcal{K}^\circ$ for some $\rho < \pi/2$.

2 Proofs

1. Proof of Proposition 1. By assumption, there exist $x_0 \in \mathcal{S}$ and $t_0 \in \mathcal{K}$ such that $x_0 \cdot t_0 = \pi/2$. The spectrum of the function $\sin(x \cdot t_0)$ consists of two points $\pm t_0 \in \mathcal{K}$, and so $\sin(x \cdot t_0) \in B_{\mathcal{K}}$.

Let $\Lambda := \{x \in \mathbb{R}^n : x \cdot t_0 \in \pi\mathbb{Z}\}$ be the zero set of $\sin(x \cdot t_0)$. Denote by $I = \{\tau x_0 : -1 \leq \tau \leq 1\} \subseteq \mathcal{S}$ the interval from $-x_0$ to x_0 . Clearly, for every point $y \in \mathbb{R}^n$ there exist $n \in \mathbb{Z}$ and $-1 \leq \tau \leq 1$ such that $y \cdot t_0 = \pi n - \tau\pi/2$. Hence, $y - \tau x_0 \in \Lambda$, which implies $\Lambda + I = \mathbb{R}^n$. \square

2. Proof of Theorem 3. We shall deduce Theorem 3 from the following

Lemma 1 *Suppose a function $g \in B_{[-\tau, \tau]}$ satisfies $|g(0)| = \|g\|_\infty$. Then*

$$|g(u)| \geq |g(0)| \cos(\tau u), \quad |u| < \pi/2\tau. \quad (5)$$

This lemma is proved in [3, proof of Theorem 4]. For completeness of presentation, we sketch the proof below.

Let us now prove Theorem 3. Take any function $f \in B_{\mathcal{K}}$. Assume first that $|f|$ attains maximum on \mathbb{R}^n , i.e. $|f(x_0)| = \|f\|_\infty$ for some $x_0 \in \mathbb{R}^n$. By (4), there exists $\lambda_0 \in \Lambda$ with $\|\lambda_0 - x_0\|_{\mathcal{K}^\circ} \leq \rho$. Consider the function of one variable $g(u) := f(x_0 + u(\lambda_0 - x_0))$, $u \in \mathbb{R}$. One may check that $g \in B_{[-\tau, \tau]}$ with $\tau = \|\lambda_0 - x_0\|_{\mathcal{K}^\circ}$. Also, clearly $|g(0)| = \|g\|_\infty$ and $g(1) = f(\lambda_0)$. Since $\tau \leq \rho < \pi/2$, we may use inequality (5) with $u = 1$:

$$\|f\|_\infty = |f(x_0)| = |g(0)| \leq \frac{|g(1)|}{\cos \tau} \leq \frac{|f(\lambda_0)|}{\cos \rho} \leq \frac{1}{\cos \rho} \|f|_\Lambda\|_\infty.$$

If $|f|$ does not attain maximum on \mathbb{R}^n , we consider the function $f_\epsilon(x) := f(x)\varphi(\epsilon x)$, where $\varphi \in B_{\epsilon\mathcal{B}}$ is any function satisfying $\varphi(0) = 1$ and $\varphi(x) \rightarrow 0$ as $|x| \rightarrow \infty$. It is clear that $f_\epsilon \in B_{\mathcal{K} + \epsilon\mathcal{B}}$ and that f_ϵ attains maximum on \mathbb{R}^n . Set $g_\epsilon(u) := f_\epsilon(x_0 + u(\lambda_0 - x_0))$, $u \in \mathbb{R}$, where x_0 and λ_0 are chosen so that $|g_\epsilon(0)| = \|f_\epsilon\|_\infty$ and $\|\lambda_0 - x_0\|_{\mathcal{K}^\circ} \leq \rho$. We have $g_\epsilon \in B_{[-\tau - \delta, \tau + \delta]}$, where $\tau = \|\lambda_0 - x_0\|_{\mathcal{K}^\circ} \leq \rho < \pi/2$ and $\delta = \delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. So, if ϵ is so small that $\tau + \epsilon < \pi/2$, we may repeat the argument above to obtain $\|f_\epsilon\|_\infty \leq \|f_\epsilon|_\Lambda\|_\infty / \cos(\rho + \delta)$. By letting $\epsilon \rightarrow 0$, we obtain (3). \square

3. Proof of Lemma 1

1. The proof in [3] is based on the following result from [4] (for some extension see [5]): *Let $f \in B_{[-\tau, \tau]}$ be a real function satisfying $-1 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$. Then for every real a the function $\cos(\tau z + a) - f(z)$ vanishes identically or else it has only real zeros. Moreover it has a zero in every interval where $\cos(\tau z + a)$ varies between -1 and 1 and all the zeros are simple, except perhaps at points on the real axis where $f(x) = \pm 1$.*

Sketch of proof. We may assume $a = 0$ and $\tau = 1$. Consider the function

$$f_\epsilon(z) := (1 - \epsilon) \frac{\sin(\epsilon z)}{\epsilon z} f((1 - \epsilon)z).$$

One may check that $f_\epsilon \in B_{[-1, 1]}$, $-1 < f(t) < 1$, $t \in \mathbb{R}$, and that the estimate holds

$$|f_\epsilon(z)| \leq \frac{e^{|y|}}{\epsilon|z|}, z = x + iy \in \mathbb{C}.$$

This shows that $|f_\epsilon(z)| < |\cos z|$ when z lies on a rectangular contour γ consisting of segments of the lines $x = \pm N\pi$, $y = \pm N$, where N is every large enough integer. By Rouché's theorem, the function $\cos z - f_\epsilon(z)$ has the same number of zeros in γ as $\cos z$, that is, $2N$ zeros. On the real axis $|f_\epsilon| \leq 1 - \epsilon$. Hence, $\cos z - f_\epsilon(z)$ is alternately plus and minus at the $2N + 1$ points $k\pi$, $|k| \leq N$, so it has $2N$ real zeros inside γ . Taking larger values of N we see that $\cos z - f_\epsilon(z)$ has exclusively real and simple zeros, which lie in the intervals $(k\pi, (k + 1)\pi)$.

The zeros of $\cos z - f(z)$ are limit points of the zeros of $\cos z - f_\epsilon(z)$ as $\epsilon \rightarrow 0$. Thus $\cos z - f(z)$ cannot have non-real zeros. Moreover, it has an infinite number of real zeros which are all simple, except those at the points $k\pi$ iff $f(k\pi) = (-1)^k$. Every interval $k\pi < z < (k + 1)\pi$ at the endpoints of which $|f(t)| < 1$ contains exactly one zero. If $f(k\pi) = (-1)^k$, we have a double zero at $k\pi$ but no further zeros in the interior or at the endpoints of the interval $((k - 1)\pi, (k + 1)\pi)$.

2. It suffices to prove Lemma 1 for real functions $f \in B_{[-\tau, \tau]}$. Since f has a local maximum at $t = 0$, the function $f(t) - \cos \tau t$ has a repeated zero at $t = 0$. By the discussion above we see that either $f(t)$ is identically equal to $\cos \tau t$ or $f(t) - \cos \tau t$ does not vanish on $[-\pi/\tau, 0) \cup (0, \pi/\tau]$. Since $|f(\pi)| \leq 1$, it follows that $f(t) > \cos \tau t$ on each of the intervals $[-\pi/\tau, 0)$ and $(0, \pi/\tau]$. \square

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